Theory of self-organized interface depinning

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We present a theoretical analysis of a recently introduced interface growth model with a global search of optimal growth sites, realized by quenched random forces. The interest in this model lies in the fact that it yields a variety of long-ranged correlated phenomena, which are characterized by scaling exponents that differ from the Kardar-Parisi-Zhang universality class [Phys. Rev. Lett. **56**, 889 (1986)], in closer correspondence with experimental observations. It is shown that all the phenomenological findings can be recovered from the theory, and all the exponents found can be computed from the knowledge of one exponent of the directed percolation problem.

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I. INTRODUCTION

Generically, the dynamical advancement of interfaces in nonlinear physics does not result in flat interfaces [1,2]. Rather, growing interfaces tend to be either unstable and develop fingers and overhangs [3], or remain self-affine, but wrinkle on a wide range of scales, with the width of the interface diverging with the size of the system. The phenomenon has attracted considerable theoretical attention; a popular model for the wrinkling of surfaces was the Kardar-Parisi-Zhang (KPZ) model [4], which is a normal form for the growth of an interface under the action of local randomness. Interestingly enough, many experiments [5,6] indicated that real interfaces do not fall under the KPZ universality class. The exponents characterizing the width of the interface differed from the prediction of the KPZ model. In addition, some detailed analysis [7] of the interface growth process indicated that the assumption about uncorrelated noise, which is fundamental to KPZ, is untenable. It became evident that one needs to understand what is the crucial ingredient that is missing in the KPZ universality class.

An interesting suggestion was made recently by Sneppen [8], who introduced a simple growth model with quenched randomness. The character of this model was chosen to represent an invasion process of the type observed in two-fluid flows in porous media [5], or in the progress of a fluid through a piece of paper [9]. In the model, the advancement of the interface is achieved by a search of a *global* minimum of the quenched noise, rather than by choosing a random point on the interface. The interface is kept self-affine by adding rules to keep the local slopes to be ± 1 (and see Sec. II for exact details). Numerical simulations indicted that this model defines a universality class which differs from KPZ, with different static (roughness) and dynamic exponents. The phenomenology of this model was explored further by Jensen and Sneppen [10], and led to the introduction of a large variety of exponents.

Theoretically, it was realized by Buldyrev *et al.* [9] and by Tang and Leschorn [11] that the roughness exponent measured by Sneppen has to do with the directed-percolation problem. In this paper we build on this connection a complete theory of the model. It turns out that the phenomenology discussed in Ref. [10], which seems quite rich, can be fully understood in terms of a simple theoretical considerations. In Sec. II we redefine the model, and elaborate on the connection to directed percolation. In Sec. III we introduce the notion of the "associated process" which we think is basic to the understanding of this model, and derive the relations between various exponents. In Sec. IV we calculate all the exponents, and in Sec. V offer concluding remarks.

II. THE MODEL AND THE CONNECTION TO DIRECTED PERCOLATION

In $1+1$ dimensions, the model is defined with a space coordinate x, $x=1,2,\ldots,L$, and with a "height" coordinate h, which is unbounded. Every lattice point (x,h) is assigned a quenched uncorrelated random number $f(x,h)$ in the interval $[0,1]$. A discrete interface $h(x)$ is grown from the initial condition $h=0$. The interface is updated in two steps: (i) finding the site with the smallest random number $f(x,h(x))$ among all the sites in front of the interface. On this site one unit is added to h, i.e., $h(x)\rightarrow h(x)+1$. (ii) After updating, the neighboring sites $y=x\pm 1$ are checked for their local slope $|h(y)-h(y\pm 1)|$. If this slope is greater than unity, the height $h(y)$ is adjusted by adding one unit. The same test is now applied to other sites, until $|h(z)-h(z\pm 1)|\leq 1$ for all points z. The last step can result in an avalanche of adjustments, and is therefore referred to as such.

The properties of this model were studied numerically [8,10]. The following is a compilation of the results obtained.

(i) The roughness exponent of the interface χ was defined by the scaling of the "width" W,

$$W\equiv\sqrt{\langle[h(x,t)-\langle h\rangle]^2\rangle}\sim L^{\chi}\ , \tag{2.1}$$

where $\langle\ \rangle$ denotes an average over space and members of

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the ensemble generated by different realizations of $f(x,h)$. Equation (2.1) is meant to hold for times t large compared to some saturation time $T_{\rm sat}$ [and see (ii)]. The numerical value of χ was found to be $\chi=0.63\pm0.02$.

(ii) A variety of temporal exponents was defined. First, the transient roughening temporal exponent $\beta_{\rm trans}$ was defined as

$$W(t)\sim t^{\beta_{\rm trans}}, \quad t<T_{\rm sat}. \tag{2.2}$$

The numerical value of $\beta_{\rm trans}$ is $\beta_{\rm trans}=0.9\pm0.1$.

Next two further exponents were defined in the saturated ensemble. The first, $\beta_{\rm crit}$, was defined through the properties of the correlation function $S(\tau)$,

$$S(\tau)\equiv\sqrt{\langle\{h(x,t+\tau)-h(x,t)-\langle[h(x,t+\tau)-h(x,t)]\rangle\}^2\rangle}\sim\tau^{\beta_{\rm crit}}, \tag{2.3}$$

where $\beta_{\rm crit}$ was found to be $\beta_{\rm crit}=0.69\pm0.02$. The second exponent, $\beta^{\infty}_{\rm crit}$, is associated with the scaling of the infinite moment of the height correlation function:

$$\langle\max_x[h(x,t+\tau)-h(x,t)]-\langle[h(x,t+\tau)-h(x,t)]\rangle\rangle\sim\tau^{\beta^{\infty}_{\rm crit}}. \tag{2.4}$$

The numerical value of $\beta^{\infty}_{\rm crit}$ is $\beta^{\infty}_{\rm crit}=0.40\pm0.05$.

(iii) Three further exponents were defined in terms of probabilities to choose a particular site x for the next growth. Suppose first that a site x_0 was chosen at time t_0. The exponent δ relates to the probability $P(t-t_0)$ that x_0 is chosen again at time t. It was found that

$$P(t-t_0)\sim(t-t_0)^{-\delta}, \tag{2.5}$$

with $\delta=0.62\pm0.03$. The probability that the first return to x_0 would be at t, $P_{\rm FR}$, scales as

$$P_{\rm FR}(t-t_0)\sim(t-t_0)^{-\gamma'}, \tag{2.6}$$

with γ' found to be $\gamma'=1.2\pm0.1$. Lastly, a very interesting quantity is the probability that after x_0 the next choice will occur at x_1, such that $|x_0-x_1|=l$. The probability for that, $P(l)$, appears to scale as

$$P(l)\sim l^{-\phi}, \tag{2.7}$$

with $\phi=2.25\pm0.05$.

(iv) Finally, the probability to find a value f in the next choice of growth site can be measured. This measurement was not performed in Ref. [10], and therefore we measured it ourselves with the result that $P(f)$ behaves like

$$P(f)\sim(f_c-f)^{\gamma-1}, \tag{2.8}$$

with $\gamma=1.9+0.1$ and $f_c=0.4615$.

The avalanches show no interesting scaling behavior. In fact, the avalanche distribution is exponential, with the mean avalanche being of the order of four sites.

Evidently, every proper theory of this model should rationalize those scaling laws and derive the numerical values of the exponents. Indeed, the roughening exponent χ has been understood already by tying the present model to directed percolation. The connection to directed percolation is explained with the help of Fig. 1. Pick a value of $f(x,h)$ which is precisely $f_c=1-\rho_c$, where ρ_c is the critical density of the directed percolation problem. Consider now the points (x,h) for which $f(x,h)\geq f_c$. Since f is uniform in the interval [0,1], the density of these points is ρ_c, and therefore they form a connected cluster (solid line in Fig. 1), which can have dangling bonds which are, however, directed. (In our thinking we choose, arbitrarily, the directedness to be form left to right.) We denote this cluster by $C_{\rm DP}(f_c)$. Consider next the subset of this cluster, which consists of all the points that belong to the unique connected path with minimal values of $h(x)$, which we denote by $B_{\rm DP}(f_c)$, see Fig. 1. The growing interface has to identify at some instant of time with a surface which for every x has a value of h which is smaller by one unit compared to $B_{\rm DP}(f_c)$ (points denoted by circles in Fig. 1). To see this, notice that any point x,h which is below this surface has a value of $f(x,h)$ lower than f_c, and will be chosen for growth before any point in $B_{\rm DP}(f_c)$. The only exception are the dangling bonds, which, however, are going to be circumvented and covered by avalanches. Thus the interface in Fig. 1 must identify with $B_{\rm DP}(f_c)$ before proceeding. In the limit $L\to\infty$ we can also state that the next point to be chosen will be a point for which $f=f_c$ *on* $B_{\rm DP}(f_c)$, and later this interface will be punctured by avalanches.

For any value of $f<f_c$ we can identify a directed-percolation cluster $C_{\rm DP}(f)$ which has a finite density of

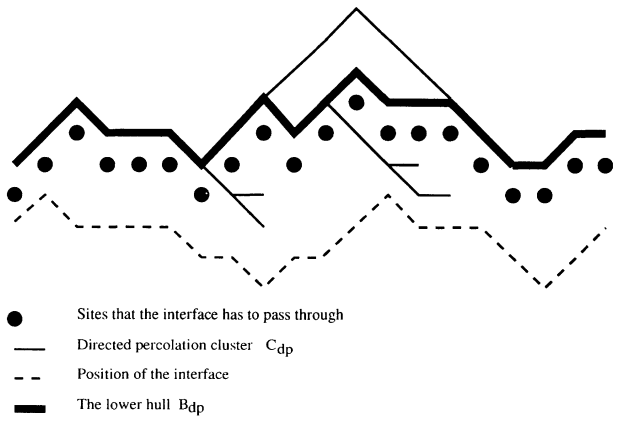

FIG. 1. A typical configuration of the growing interface with respect to the blocking directed-percolation interface $C_{\rm DP}$ and its lower hull $B_{\rm DP}$. All solid lines pertain to the $C_{\rm DP}$. The bold line is $B_{\rm DP}$. The interface has to identify with the path denoted by the dotted sites before it penetrates through $B_{\rm DP}$.

points $\rho(f)$, which increases when f is removed from f_c. Considering a situation where the highest value of the quenched noise in front of the interface is smaller than some f, we can define a blocking surface $B_{\rm DP}(f)$. This surface is defined as the lowest connected path belonging to $C_{\rm DP}(f)$. Evidently, the interface has to identify on its way with paths removed down one unit in h for each and every such cluster $B_{\rm DP}(f)$. For f not too far from f_c we can use the results for the transverse and parallel correlation lengths for $C_{\rm DP}$, i.e.,

$$\xi_\perp \sim |\rho-\rho_c|^{-1.097} , \tag{2.9a}$$

$$\xi_\parallel \sim |\rho-\rho_c|^{-1.733} , \tag{2.9b}$$

to estimate for $B_{\rm DP}(f)$ a roughening exponent

$$\xi_\perp \sim \xi_\parallel^{1.097/1.733} = \xi_\parallel^{0.633} . \tag{2.10}$$

This is the justification for the measured value of χ. We should stress that in deriving (2.10) we assumed that the geometric exponents of $B_{\rm DP}(f)$ and $C_{\rm DP}(f)$ are the same. This is a reasonable assumption, but we are aware of no proof for its validity.

All the other exponents in this problem need some more thinking. We turn now to the fundamental process which we call the "associated process," whose properties will yield eventually all the desired information.

III. THE "ASSOCIATED PROCESS" AND ITS SCALING PROPERTIES

A. Definition of the associated process

Suppose that at time $t=t_0$ the point $x_0,h(x_0)$ was chosen for growth, and that $f(x_0,h(x_0))\geq f_0$. Define the f_0 associated process, denoted as $\mathcal{P}_{f_0}(s)$, as the series of steps in which the points $x_1,x_2,\ldots,x_s$ were chosen for growth, if the following conditions are met:

$$f(x_1)<f_0,\quad f(x_2)<f_0,\ldots,\quad f(x_s)<f_0 , \tag{3.1a}$$

$$f(x_{s+1})\geq f_0 . \tag{3.1b}$$

We can state a few properties of the associated processes.

(i) In the limit $L\to\infty$ there do not exist associated processes with $f_0>f_c$. The reason for this fact can be found in the discussion in the last section. The blocking surface $B_{\rm DP}(f_c)$ is going to be punctured exactly at f_c in the limit $L\to\infty$.

(ii) The set of points $x_1,x_2,\ldots,x_s$, together with the points exposed by avalanches, is simply connected and compact. The reason for this is that the associated process is delineated by two blocking surfaces, and there is no way to jump to a region that is not enclosed by these lines. The point x_0 belongs to the lower bounding line, and has the minimal value of f on that line. Accordingly, the definition of the associated process guarantees that no jump outside the bounded region will occur. (iii) A consequence of (i) and (ii) is that given a value of $f_0<f_c$, the union of the associated processes. $\mathcal{P}_{f_0}(s)$ is the lattice x,h. In other words, every value of f_0 defines a network in which the strings are given by the blocking surfaces characterized by f_0, and the holes in the net are the associated processes, with points $\{x_j\}_{j=1}^s$ plus avalanches. It should be stressed that this network is *not* the same as the network obtained by a directed-percolation cluster of points with a density $\rho>\rho_c$. The reason for the difference is summarized in Fig. 2, and is explained in the figure caption to Fig. 2.

(iv) Define $K_{f_0}(s)$ to be the distribution of s values in the union of f_0 associated processes. We claim (and demonstrate below) that this distribution has the scaling form

$$K_{f_0}(s)=s^{-\tau}g(s/\Delta f^{-\nu}) , \tag{3.2}$$

where $\Delta f=f_c-f_0$, and τ and ν are scaling exponents. Since (3.2) is used repeatedly below, we shall pause now to show that the exponents τ and ν are not independent exponents, but are actually derivable from the knowledge of the roughness exponent χ.

B. Calculation of τ and ν in terms of the roughness exponent

To calculate τ, we use the fact that every hole in the f_0 network is characterized by a height $r_\perp$ and a width $r_\parallel$. Evidently, the number of steps s in the associated process defining the hole is of the order

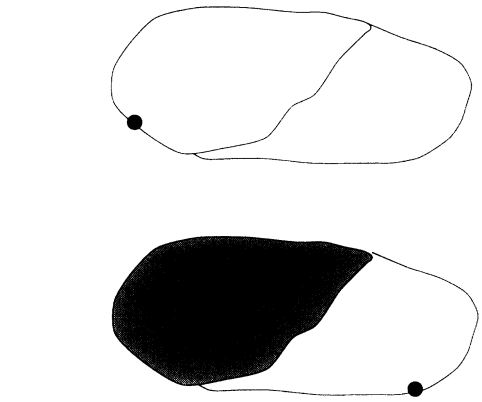

FIG. 2. A diagrammatic explanation of the difference between the network of directed percolation and the present interface growth model. The diagrams show typical two holes in a directed-percolation network. In our process the presence of one or two holes in the network is determined by the position of the initial site of the associated process. Starting as in the upper diagram the process will eliminate the middle path by avalanches, and the associated process will cover both holes. In the lower diagram the middle path acts as a blocking surface and both holes will host different associated processes. This dependence on the dynamical history is one important reason for the different scaling exponents.

$$s \sim r_\perp r_\parallel , \tag{3.3}$$

as long as the avalanches have a mean size which is independent of s [12]. Since every hole is trapped between two paths belonging to $B_{\rm DP}(f_0)$, we expect that

$$r_\perp \sim r_\parallel^{\chi} , \tag{3.4a}$$

$$s \sim r_\parallel^{1+\chi} . \tag{3.4b}$$

To proceed, introduce now the distributions of $r_\perp$ and $r_\parallel$, denoted as $K_{f_0}(r_\perp)$ and $K_{f_0}(r_\parallel)$, respectively. We can write for $K_{f_0}(r_\parallel)$, in correspondence with (3.2), in the scaling form

$$K_{f_0}(r_\parallel)=r_\parallel^{-\tau_\parallel}\tilde{g}\left[\frac{r_\parallel}{\Delta f^{-\nu_\parallel}}\right] . \tag{3.5}$$

On the one hand, we can deduce the value of $\tau_\parallel$ from the geometry of the network. Every hole in the network is bordered by two paths whose roughening exponent is χ. Thus the distribution of $r_\parallel$ should be like the distribution of intervals between the intersection of the graph with a given horizontal axis. Since the dimension of a graph with roughening (Hurst) exponent χ is $2-\chi$, the dimension of the intersection points is $1-\chi$. Therefore the number of intersecting points in an interval of length R scales as $R^{1-\chi}$, and the mean interval size $\langle r_\parallel \rangle$ scales as R^{χ}. Using Eq. (3.5) we see that

$$R^{\chi} \sim \int_{r_\parallel}^{R} -\tau_\parallel r_\parallel dr_\parallel . \tag{3.6a}$$

This determines $\tau_\parallel$ as

$$\tau_\parallel = 2-\chi . \tag{3.6b}$$

We tested this prediction in numerical simulations and found $\tau_\parallel = 1.37 \pm 0.05$, in agreement with the value of χ.

On the other hand, we can invoke the connection between $K_{f_0}(r_\parallel)$ and $k_{f_0}(s)$, which is given by

$$K_{f_0}(r_\parallel)=K_{f_0}(s)\frac{ds}{dr_\parallel}=s^{-\tau}g\left[\frac{r_\parallel^{1+\chi}}{\Delta f^{-\nu}}\right]r_\parallel^{\chi} . \tag{3.7}$$

Using (3.4b) in (3.7) yields

$$\begin{aligned} K_{f_0}(r_\parallel) &= r_\parallel^{-\tau(1+\chi)+\chi} g\left[\frac{r_\parallel^{1+\chi}}{\Delta f^{-\nu}}\right] \\ &= r_\parallel^{-\tau(1+\chi)+\chi}\tilde{g}\left[\frac{r_\parallel}{\Delta f^{-\nu/(1+\chi)}}\right] . \end{aligned} \tag{3.8}$$

Comparing (3.8) and (3.5) we deduce that $\tau_\parallel=\tau(1+\chi)-\chi$ and $\nu_\parallel=\nu/(1+\chi)$. Using (3.6b) we find finally

$$\tau=\frac{2}{1+\chi} . \tag{3.9}$$

In simulations we got $\tau=1.23\pm0.03$, in good agreement with (3.9).

To calculate the exponent ν of (3.2) we need to estimate the changes in the network which are caused by changing the value of Δf. Using Eq. (3.2) to calculate the mean of s, we find

$$\langle s \rangle_{f_0} \sim \Delta f^{-\nu(2-\tau)} . \tag{3.10}$$

The number of holes in the f_0 network, denoted n_{f_0}, is of the order of the number of lattice points over $\langle s \rangle_{f_0}$:

$$n_{f_0} \sim \Delta f^{\nu(2-\tau)} . \tag{3.11}$$

When we change f_0 by a differential df (i.e., $f_0 \to f_0+df$), there is a differential change dn_{f_0} in the number of holes, since some boundaries in the fishnet disappear, and some holes merge to create bigger holes. The probability to eliminate a site on the fishnet is proportional to the total length of the net, which we denote by L_{f_0}. Denote also the typical numbers of holes which disappear when one boundary is broken by $\tilde{N}$. Since we assume that the fishnet remains geometrically similar at different values of f_0 (only with different size holes) we can assert that $\tilde{N}$ is independent of f_0. Accordingly we write

$$\frac{dn_{f_0}}{df}=L_{f_0}\tilde{N} . \tag{3.12}$$

On the other hand, L_{f_0} can be easily estimated as

$$L_{f_0} \sim n_{f_0}[\langle r_\perp \rangle + \langle r_\parallel \rangle] \sim n_{f_0}\langle r_\parallel \rangle , \tag{3.13}$$

where the last step stems from the fact that the holes are elongated (i.e., $\chi<1$). Of course, $\langle r_\parallel \rangle$ can be computed from (3.5),

$$\langle r_\parallel \rangle \sim \Delta f^{-\nu_\parallel(2-\tau_\parallel)}=\Delta f^{-\nu_\parallel \chi} . \tag{3.14}$$

Combining (3.11)–(3.14) we get the scaling relation $\nu\chi-1=\nu\chi-\nu_\parallel\chi$, or $\nu_\parallel=1/\chi$. Since $\nu_\parallel=\nu/(1+\chi)$ we can state finally that

$$\nu=(1+\chi)/\chi . \tag{3.15}$$

Equations (3.2), (3.9), and (3.15) complete the identification of the scaling properties of the associated processes. We stress that the values of τ and ν are different from the corresponding exponents of the directed-percolation problem. The reasons are those discussed in Fig. 2 and the statement of similarity of the network at different values of f_0. We turn now to using these results to derive the phenomenology of the model.

IV. DERIVATION OF THE SCALING EXPONENTS

A. Steady-state exponents

We begin with the probability distribution (2.8) to find a value of f of the quenched noise in the next growth site. From the definition of the associated process $\mathcal{P}_f(s)$ we know that, given that at time t_0 we hit a value larger than f (i.e., between f and f_c), then next we can either hit a site with a yet larger value of the noise, or start an associated process. In the second case we know that for s consecutive steps we shall see only values of the noise smaller than f. If f is very close to f_c the first possibility carries a very small weight. Therefore for the sake of our argu-

ment we can say that the probability to see any f^* in the interval $f_c - f$ depends inversely on $\langle s \rangle_f$,

$$\text{Prob}\{f \le f^* < f_c\} \sim \langle s \rangle_f^{-1} \sim (f_c - f)^{\nu(2-\tau)} \sim (f_c - f)^2 , \quad (4.1)$$

where (3.9) and (3.15) have been used. Next, since the events of hitting in the interval (f, f_c) are statistically uncorrelated, the probability to start a $\mathcal{P}_f$ process by *exactly* hitting f is $(f_c - f)^{-1}$. Combining this with (4.1) we conclude that

$$P(f) \sim (f_c - f) , \quad (4.2)$$

or, in other words, the exponent γ of Eq. (2.8) is simply

$$\gamma = 2 . \quad (4.3)$$

This result agrees well with our numerical simulations.

Consider next the dynamical exponents summarized in Sec. II. The growth process is composed of a succession of associated processes which are local and compact, with infrequent big jumps too far away associated processes. In a typical local process a hole of size $r_\perp r_\parallel$ gets covered in some time interval of time $\Delta t = t - t_0$:

$$r_\perp r_\parallel \sim \Delta t . \quad (4.4)$$

Since we remember that $r_\perp \sim r_\parallel^\chi$, we conclude that

$$r_\parallel \sim \Delta t^{1/(1+\chi)} , \quad (4.5a)$$

$$r_\perp \sim \Delta t^{\chi/(1+\chi)} . \quad (4.5b)$$

First we notice that (4.5b) describes the rate of growth of the maximal perpendicular length, and therefore the exponent should be the same as $\beta_{\text{crit}}^{\infty}$ of Eq. (2.4):

$$\beta_{\text{crit}}^{\infty} = \frac{\chi}{1+\chi} = 0.388\ldots , \quad (4.6)$$

in excellent agreement with the numerical finding.

The probability that the same point x that was chosen at time t_0 is chosen again at time $t_0 + \Delta t$ is simply proportional to $r_\perp / \Delta t$, and therefore $P(t - t_0)$ of Eq. (2.5) reads

$$P(t - t_0) \sim \Delta t^{\chi/(1+\chi)-1} . \quad (4.7)$$

Thus the exponent δ satisfies

$$\delta = \frac{1}{1+\chi} = 0.612\ldots \quad (4.8)$$

again in agreement with the findings.

To calculate β_{crit} of Eq. (2.3) we need to be a bit more delicate, since we need to average over an ensemble of different contributions arising from different associated processes. The easiest way to think about this average is to say that $|h(x,t+\tau) - h(x,t)| \sim r_\perp \sim \Delta t^{\chi/(1+\chi)}$, and when we average over x we need to weigh each such contribution by its "length" $r_\parallel \sim \Delta t^{1/(1+\chi)}$. Thus $S(L,\tau)$ of Eq. (2.3) is expected to scale as $\sqrt{r_\perp^2 r_\parallel}$, or

$$S(L,\tau) \sim \tau^{(1+2\chi)/2(1+\chi)} . \quad (4.9)$$

Consequently

$$\beta_{\text{crit}} = \frac{1+2\chi}{2(1+\chi)} = 0.694\ldots \quad (4.10)$$

in agreement with the numerical result.

Finally, we want to estimate the exponent ϕ of Eq. (2.7). We shall only derive here an upper bound on ϕ and will see that it is close enough to the numerical result. To do this, think about the interface growth process as a succession of associated processes, each one characterized by a value of Δf. As long as the associated process continues, the jumps l between successive chosen sites are relatively small. Once the associated process is finished, we jump a distance which is *at least* of the order of $r_\parallel$ of the associated process. Thus, for a given Δf the jump distribution can be estimated as (3.5) where we remember that the actual jumps can be *larger* than $r_\parallel$. The distribution $P(l)$ will be estimated from an average over all values of Δf, i.e.,

$$P(l) \sim \int_0^{1-p_c} P(f = f_c - \Delta f) K_f(l) d\Delta f = \int_0^{1-p_c} \Delta f^{\gamma-1} l^{-\tau_\parallel} \tilde{g}\left[\frac{l}{\Delta f - \nu_\parallel}\right] d\Delta f . \quad (4.11)$$

Comparing with (2.7) we conclude that

$$\phi = \tau_\parallel = \gamma/\nu_\parallel = 2 + \chi . \quad (4.12)$$

Indeed, the numerical estimate is too high, 2.63 instead of 2.25, as we expected. The jumps can be larger than our estimate in (4.11), leading to a weaker fall-off in $P(l)$.

B. Transient exponents

One of the interesting observations of Ref. [10] was that the transient dynamical exponent was different from its steady-state value. This is in sharp difference with models of the KPZ class, but seems to be a common feature of models with quenched noise. To calculate β_{trans} we first establish a physical picture of the initial organization of the interface.

Starting from a flat interface, we begin to advance the interface in a succession of local compact processes, which have a typical scale that grows with time. Denote the maximal length scale in the system, which is the length of the maximal compact domain constructed until time t by ξ_t. As long as ξ_t is much smaller than the size of the system L, the domain size distribution is going to be the same as in Eq. (3.5), but with a time-dependent cutoff:

$$K_t(r_\parallel) = r_\parallel^{-\tau_\parallel} \tilde{g}\left[\frac{r_\parallel}{\xi_t}\right] . \quad (4.13)$$

After some short maturation time, the total length of the domains is of $O(L)$. The average domain length is computed from (4.13), and it reads $\langle r_\parallel \rangle_t \sim \xi_t^\chi$. The number of domains grown until time t is therefore L/ξ_t^χ and we denote it by N_t. Obviously, the total number of affected sites at time t is proportional to t. Accordingly,

$$t \sim N_t \int_1^{\xi_t^{\chi+1}} s s^{-\tau} ds , \quad (4.14)$$

where the upper bound is the size of the maximal domain. From this we get that ξ_t satisfies the relation

$$\xi_t \sim t^{1/\chi} . \tag{4.15}$$

The maximal width W_t should be of the order of $\xi_t{}^{\chi}$, and therefore

$$\beta_{\text{trans}} = 1 \tag{4.16}$$

in agreement with the numerics.

We should notice that the exponent characterizing ξ_t, cf. Eq. (4.15), is an interesting prediction that can be measured. An additional prediction, which is even more interesting, relates to the approach to criticality in the organization process of the interface. We shall assert that the growth of ξ_t is a sign of the tendency to choose f values that approach f_c more and more. In time, the maximal observed f should approach f_c as a power law in time, $\Delta f_t \sim t^{\zeta}$. Using the previous relation between the typical size $r_{\parallel}$ and Δf we can estimate

$$\xi_t \sim \Delta f_t{}^{-\nu_{\parallel}} \tag{4.17}$$

or

$$\Delta f_t \sim t^{-1/\nu_{\parallel}} = t^{-\chi} . \tag{4.18}$$

This prediction has not been tested yet.

V. CONCLUSIONS

The main lesson to be learned is that the growth process described above is fundamentally different from the KPZ class. The latter is initiated by a random force which can hit at arbitrary points on the lattice. In the present model the growth is a successive series of localized compact invasions interrupted by rare jumps between them [13]. For this reason the concept of the associated process is so fundamental to the understanding of the scaling properties of this model. This is also the reason for the difference between the transient and the saturated dynamical exponents.

Although formally the observation that β_{crit} differs from $\beta_{\text{crit}}^{\infty}$ might qualify as "multiscaling" we stress that they are both derivable from the knowledge of one independent exponent, χ. In fact, one could introduce exponents β_q for q-order structure functions, and observe a q dependence. This is a rather trivial case of multiscaling since the exponents do not require independent information for their evaluation.

It appears that the wealth of predictions available now for the scaling properties of this model suffices to test the applicability of this model to experimental situations that fall outside the KPZ universality class.

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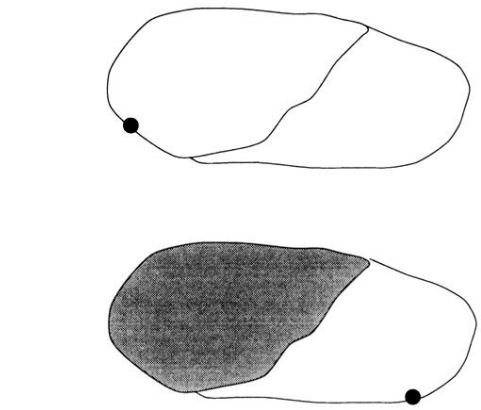

FIG. 2. A diagrammatic explanation of the difference between the network of directed percolation and the present interface growth model. The diagrams show typical two holes in a directed-percolation network. In our process the presence of one or two holes in the network is determined by the position of the initial site of the associated process. Starting as in the upper diagram the process will eliminate the middle path by avalanches, and the associated process will cover both holes. In the lower diagram the middle path acts as a blocking surface and both holes will host different associated processes. This dependence on the dynamical history is one important reason for the different scaling exponents.